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ROCKET BOOSTER CONTROL

SECTION 4

A SUFFICIENT CONDITION

IN OPTIMAL CONTROL

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ROCKET BOOSTER CONTROL.

SECTION 4:

A SUFFICIENT CONDITION IN OPTIMAL CONTROL

(NASA Contract NASw-563)

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FOREWORD

This document is one of sixteen sections that comprise the final report prepared by the Minneapolis-Honeywell Regulator Company for the National Aeronautics and Space Administration under contract NASw-563. The report is issued in the following sixteen sections to facilitate updating as progress warrants:

1541-TR 1 Summary

1541-TR 2 Control of Plants Whose Representation Contains Derivatives of the Control Variable

1541-TR 3 Modes of Finite Response Time Control

1541-TR 4 A Sufficient Condition in Optimal Control

1541-TR 5 Time Optimal Control of Linear Recurrence Systems

1541-TR 6 Time-Optimal Bounded Phase Coordinate Control of Linear Recurrence Systems

1541-TR 7 Penalty Functions and Bounded Phase Coordinate Control

1541-TR 8 Linear Programming and Bounded Phase Coordinate Control

1541-TR 9 Time Optimal Control with Amplitude and Rate Limited Controls

1541-TR 10 A Concise Formulation of a Bounded Phase Coordinate Control Problem as a Problem in the Calculus of Variations

1541-TR 11 A Note on System Truncation

1541-TR 12 State Determination for a Flexible Vehicle Without a Mode Shape Requirement

1541-TR 13 An Application of the Quadratic Penalty Function Criterion to the Determination of a Linear Control for a Flexible Vehicle

1541-TR 14 Minimum Disturbance Effects Control of Linear Systems with Linear Controllers

1541-TR 15 An Alternate Derivation and Interpretation of the Drift-Minimum Principle

1541-TR 16 A Minimax Control for a Plant Subjected to a Known Load Disturbance

Section 1 (1541-TR 1) provides the motivation for the study efforts and objectively discusses the significance of the results obtained. The results of inconclusive and/or unsuccessful investigations are presented. Linear programming is reviewed in detail adequate for sections 6, 8, and 16.

It is shown in section 2 that the purely formal procedure for synthesizing an optimum bang-bang controller for a plant whose representation contains derivatives of the control variable yields a correct result.

In section 3 it is shown that the problem of controlling m components $(1 < m \leq n)$, of the state vector for an n-th order linear constant coefficient plant, to zero in finite time can be reformulated as a problem of controlling a single component.

Section 4 shows Pontriagin's Maximum Principle is often a sufficient condition for optimal control of linear plants.

Section 5 develops an algorithm for computing the time optimal control functions for plants represented by linear recurrence equations. Steering may be to convex target sets defined by quadratic forms.

In section 6 it is shown that linear inequality phase constraints can be transformed into similar constraints on the control variables. Methods for finding controls are discussed.

Existence of and approximations to optimal bounded phase coordinate controls by use of penalty functions are discussed in section 7.

In section 8 a maximum principle is proven for time-optimal control with bounded phase constraints. An existence theorem is proven. The problem solution is reduced to linear programming.

A backing-out-of-the-origin procedure for obtaining trajectories for time-optimal control with amplitude and rate limited control variables is presented in section 9.

Section 10 presents a reformulation of a time-optimal bounded phase coordinate problem into a standard calculus of variations problem.

A mathematical method for assessing the approximation of a system by a lower order representation is presented in section 11.

Section 12 presents a method for determination of the state of a flexible vehicle that does not require mode shape information.

The quadratic penalty function criterion is applied in section 13 to develop a linear control law for a flexible rocket booster.

In section 14 a method for feedback control synthesis for minimum load disturbance effects is derived. Examples are presented.

Section 15 shows that a linear fixed gain controller for a linear constant coefficient plant may yield a certain type of invariance to disturbances. Conditions for obtaining such invariance are derived using the concept of complete controllability. The drift minimum condition is obtained as a specific example.

In section 16 linear programming is used to determine a control function that minimizes the effects of a known load disturbance.

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A SUFFICIENT CONDITION IN OPTIMAL CONTROL*

By E. B. Lee‡

ABSTRACT

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A theorem is proven which covers most of the known cases where L. S. Pontriagins' Maximum Principle is a sufficient, as well as a necessary, condition for optimal control. AUTHOR

INTRODUCTION

Consider the system

$$\dot{x}^o = f^o(x,t) + h^o(u,t) \tag{1}$$

$$\dot{x} = A(t)x + h(u,t) \tag{2}$$

with $x(t_o) = x_o$ and $x^o(t_o) = 0$. Here f^o, h^o, A, and h are continuous in all arguments. x is the system state, an n vector; u is the control, an m vector. x^o is a scalar variable which measures the quality of control.

If u(s) is any control function on the interval $[t_o,t]$ we will write the corresponding response of equations (1) and (2) as $\hat{x}_u(t) = (x_u^o(t), x_u(t))$. The control u is restricted to a set $\Omega \subset R^m$. It is assumed that either Ω is compact or that h, and h^o are such that

$$\max_{u\epsilon\Omega}\{\lambda\cdot h(u,t) + \lambda^o h^o(u,t)\}$$

exists for each $t\epsilon[t_o,T]$ and $\hat{\lambda} = (\lambda^o,\lambda)\epsilon R^{n+1}$ with $\lambda^o < 0$.

It is further assumed that $f^o(x,t)$ is a single-valued, convex function of x for each $t\epsilon[t_o,T]$, that is,

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* Prepared under contract NASw-563 for the NASA

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$$\frac{\partial f^o}{\partial x}(x,t)\cdot(\omega-x) + f^o(x,t) \leq f^o(\omega,t)$$

for all x and $\omega \epsilon R^n$ and $t\epsilon[t_o,T]$.

The following definitions are needed:

DEFINITION 1. $u^*(s)$ on $[t_o,T]$ is an extremal control if there exists $\lambda(t_o) = (\lambda_o^1,\lambda_o^2,\ldots,\lambda_o^n)$ and λ^o = constant < 0, such that

$$\lambda^o h^o(u^*(s),s) + \lambda(s)\cdot h(u^*(s),s) = \max_{u\epsilon\Omega}\{\lambda^o h^o(u,s) + \lambda(s)\cdot h(u,s)\}$$

with

$$\dot{\lambda} = - A'(t)\lambda - \lambda^o \frac{\partial f^o}{\partial x}(x_{u^*}(t),t), \quad (' \text{ denotes transpose}).$$

Here $x_{u^*}(t)$ is the response corresponding to $u^*(s)$, $t_o \leq s \leq t \leq T$.

DEFINITION 2. The control $u(s)$ is allowable if it is a measurable real valued vector function with range in Ω on $[t_o,T]$.

DEFINITION 3. The set of attainability $K(T,x_o)$ is the collection of end points of the responses $\hat{x}_u(t) = (x_u^o(t),x_u(t))$ which initiate at $(0,x_o)$ for all allowable controls $u(s)$ on $[t_o,T)$.

The problem of optimal control studied here is to select allowable controls $u(s)$ which "steer" the response $x_u(t)$ from the initial point x_o at time t_o to a prescribed target set G at time $T < \infty$ and minimize the cost functional of control $C(u) = g(x(T)) + x^o(T)$. Here g is a continuously differentiable function of x. An allowable control which provides an absolute minimum for $C(u)$ amongst the set of all allowable controls which steer $x_u(t)$ from x_o to G is called an optimal control. Note, the free end point problem results when $G = R^n$.

DEVELOPMENT

A basic inequality is established and then the main theorem is proven.

LEMMA. Let $u^*(s)$, $t_o \leq s \leq T$, be an allowable extremal control with corresponding response $\hat{x}_{u^*}(t)$ which initiates at $\hat{x}(t_o)=(0,x(t_o))$, then $\hat{\lambda}(T)\cdot\hat{x}_{u^*}(T)\geq\hat{\lambda}(T)\cdot\hat{\omega}$ for $\lambda^o<0$ and all $\hat{\omega}\in K(T,x_o)$.

PROOF. Consider

$$\frac{d(\hat{\lambda}\cdot\hat{x}_{u^*})}{dt} = \lambda^o\dot{x}^o_{u^*} + \dot{\lambda}\cdot x_{u^*} + \lambda\cdot\dot{x}_{u^*}$$

$$= \lambda^o(f^o(x_{u^*},t) + h^o(u^*,t)) +$$

$$(-A'(t)\lambda - \lambda^o\frac{\partial f^o}{\partial x}(x_{u^*},t))\cdot x_{u^*}$$

$$+ \lambda\cdot(A(t)x_{u^*} + h(u^*,t)).$$

Upon integrating both sides between t_o and T we obtain

$$\lambda^o x^o_{u^*}(T) + \lambda(T)\cdot x_{u^*}(T) - \lambda(t_o)\cdot x(t_o) =$$

$$\int_{t_o}^{T}\{\lambda^o(f^o(x_{u^*}(t),t) - \frac{\partial f^o}{\partial x}(x_{u^*}(t),t)\cdot x_{u^*}(t))$$

$$+ \lambda^o h^o(u^*(t),t) + \lambda(t)\cdot h(u^*(t),t)\}dt$$

Let $x_u(t)$ be any other response with initial value $x_o = x(t_o)$ for which we calculate

$$\lambda^o x^o_u(T) + \lambda(t)\cdot x_u(T) - \lambda(t_o)\cdot x(t_o) =$$

$$\int_{t_o}^{T}\{\lambda^o(f^o(x_u(t),t) - \frac{\partial f^o}{\partial x}(x_{u^*}(t),t)\cdot x_u(t))$$

$$+ \lambda^o h^o(u(t),t) + \lambda(t)\cdot h(u(t),t)\}dt.$$

But $\lambda^o h^o(u^*(t),t) + \lambda(t)\cdot h(u^*(t),t) \geq \lambda^o h^o(u(t),t) + \lambda(t)\cdot h(u(t),t)$.

Thus if $\lambda^o(f^o(x_{u*}(t),t) - \frac{\partial f^o}{\partial x}(x_{u*}(t),t)\cdot x_{u*}(t)) \geq \lambda^o(f^o(x_u(t),t) - \frac{\partial f^o}{\partial x}(x_{u*}(t),t)\cdot x_u(t))$

we obtain the desired inequality. This is certainly true if $\lambda^o \leq 0$ and

$$\frac{\partial f^o}{\partial x}(x_{u*}(t),t)\cdot x_{u*}(t) - f^o(x_{u*}(t),t) \geq \frac{\partial f^o}{\partial x}(x_{u*}(t),t)\cdot x_u(t) - f^o(x_u(t),t),$$

which is the convexity condition on f^o.

Thus we have

$\lambda^o x^o_{u*}(T) + \lambda(T)\cdot x_{u*}(T) \geq \lambda^o x^o_u(T) + \lambda(T)\cdot x_u(T)$ or

$\hat{\lambda}(T)\cdot\hat{x}_{u*}(T) \geq \hat{\lambda}(T)\cdot\hat{\omega}$ all $\hat{\omega}\in K(T,x_o)$ and the lemma is established.

The basic inequality of the lemma enables us to establish the sufficiency of the maximum principle in a number of cases. These results are summarized as a theorem:

THEOREM.

A) Consider the cost functional of control $C(u) = x^o(T)$ and as target set G a point x_1. Let $u^*(s)$, $t_o \leq s \leq T$, be an allowable extremal control which steers the corresponding response $x_{u*}(t)$ from x_o at t_o to x_1 at T, then $u^*(s)$ is an optimal control.

B) Consider the cost functional $C(u) = g(x(T)) + x^o(T)$ with $g(x)$ a convex function of x and consider the target set $G = R^n$, (this is the free end point problem). Let $x(t_o) = x_o$. Then $u^*(s)$, $t_o \leq s \leq T$, is an optimal control if it is an allowable extremal control with $\hat{\lambda}(T) = (-1, -\frac{\partial g}{\partial x}(x_{u*}(T)))$, (The condition on $\hat{\lambda}(T)$ is called a transversality condition).

C) Consider the cost functional $C(u) = x^o(T)$ and the convex, closed, target set $G = \{x | \gamma(x) \leq c\} \subset R^n$, where γ is differentiable and c a constant. Let $u^*(s)$, $t_o \leq s \leq T$, be an allowable extremal control which steers $x_{u^*}(t)$ from x_o at t_o to $x_1 \epsilon G$ at T with $\lambda(T)$ an interior normal[‡] to G at $x_{u^*}(T)$ on ∂G, then $u^*(s)$ is optimal if such a control exists. (If there is no such $u^*(s)$ then the minimum may occur interior to G in which case B) applies with $\hat{\lambda}(T) = (-1,0,0...0)$ and if G is just one point part A) is obtained.)

PROOF:

A) From the lemma

$$\hat{\lambda}(T) \cdot \hat{x}_{u^*}(T) \geq \hat{\lambda}(T) \cdot \hat{\omega} \text{ for } \lambda^o < o \text{ all } \hat{\omega} \epsilon K(T, x_o).$$

Thus

$$\lambda(T) \cdot x_{u^*}(T) + \lambda^o x^o_{u^*}(T) \geq \lambda(T) \cdot x_u(T) + \lambda^o x^o_u(T).$$

But, comparing only those responses that end at x_1, that is, those for which $x_{u^*}(T) = x_u(T) = x_1$, the basic inequality becomes

$$\lambda^o x^o_{u^*}(T) \geq \lambda^o x^o_u(T).$$

Since $\lambda^o < 0$ we have $C(u^*) = x^o_{u^*}(T) \leq x^o_u(T) = C(u)$ and therefore $u^*(s)$ is optimal.

B) With $\hat{\lambda}(T) = (-1, \frac{-\partial g}{\partial x}(x_{u^*}(T)))$ the inequality of the lemma is

$$\frac{-\partial g}{\partial x}(x_{u^*}(T)) \cdot x_{u^*}(T) - x^o_{u^*}(T) \geq \frac{-\partial g}{\partial x}(x_{u^*}(T)) \cdot x_u(T) - x^o_u(T)$$

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‡ λ is an interior normal to G at x on ∂G if λ is orthogonal to a support plane of G at x and is directed into the halfspace containing G. Thus G need not have an interior to have interior normals. Note if G does not have an interior we can still approximate it by a $\gamma(x)$.

Adding and subtracting $g(x_{u^*}(T))$ on the left side and $g(x_u(T))$ on the right side the last inequality becomes

$$-x^o_{u^*}(T) - g(x_{u^*}(T)) + g(x_{u^*}(T)) - \frac{\partial g}{\partial x}(x_{u^*}(T))\cdot x_{u^*}(t) \geq$$

$$-x^o_u(T) - g(x_u(T)) + g(x_u(T)) - \frac{\partial g}{\partial x}(x_{u^*}(T))\cdot x_u(T).$$

But,

$$\frac{\partial g}{\partial x}(x_{u^*}(T))\cdot(x_{u^*}(T) - x_u(T)) + g(x_u(T)) \geq g(x_{u^*}(T))$$

if g is a convex function of x.

Therefore

$$-C(u^*) = -x^o_{u^*}(T) - g(x_{u^*}(T)) \geq$$
$$-x^o_u(T) - g(x_u(T)) = -C(u),$$

or $C(u^*) \leq C(u)$. Hence part B) is established.

C) Assume for simplicity that γ was picked to be a convex function on G with $\partial G = \{x \mid \gamma(x) = c\}$. Consider only boundary points $x_{u^*}(T)$ where it is required that $\lambda(T) = k\{-\frac{\partial \gamma}{\partial x}(x_{u^*}(T))\}$ in order for $\lambda(T)$ to be an interior normal to G at $x_{u^*}(T)$ on ∂G, (let $k = 1$).

The inequality of the lemma can then be written

$$\lambda(T)\cdot x_{u^*}(T) = \lambda^o x^o_{u^*}(T) + \{\frac{-\partial \gamma}{\partial x}(x_{u^*}(T))\}\cdot x_{u^*}(T) \geq$$

$$\lambda(T)\cdot x_u(T) = \lambda^o x^o_u(T) + \{\frac{-\partial \gamma}{\partial x}(x_{u^*}(T))\}\cdot x_u(T)$$

If $x_{u^*}(T)$ in on the boundary of G it is also true that $\gamma(x_{u^*}(T)) = c \geq \gamma(x_u(T))$ for all allowable responses $x_u(T) \epsilon G$. Adding the last two inequalities we obtain

$$\lambda^o x^o_{u^*}(T) - \frac{\partial \gamma}{\partial x}(x_{u^*}(T))\cdot x_{u^*}(T) + \gamma(x_{u^*}(T)) \geq$$
$$\lambda^o x^o_u(T) - \frac{\partial \gamma}{\partial x}(x_{u^*}(T))\cdot x_u(T) + \gamma(x_u(T)).$$

But again if $x_{u^*}(T)$ is on ∂G and $x_u(T)$ is in the convex set G we

have

$$\frac{\partial \gamma}{\partial x}(x_{u*}(T)) \cdot [x_{u*}(T) - x_u(T)] + \gamma(x_u(T)) \geq \gamma(x_{u*}(T))$$

and therefore

$$C(u^*) \leq C(u). \quad \text{Q.E.D.}$$

REMARKS

If the set of attainability is closed there will exist optimum control provided there is at least one control that steers the response to the desired end point $x_1 \epsilon G$, assuming G is also closed. The property of closure is discussed in reference 2 in which a bibliography and discussion of cases are presented. The set of attainability is also known to be closed if

$h(u,t) = B(t)u$, $f^o(x,t) = x \cdot W(t)x$ and $h^o(u,t) = u \cdot U(t)u$, for $W(t)$, $U(t)$ positive definite on $[t_o, T]$.

When the set of attainability is closed, in the above case, the inequality of the lemma establishes that its lower (exterior normal with $\lambda^o < 0$) surface is convex. For if it was otherwise we would be led to a contradiction of the maximum principle. Note that the transversality condition (reference 1) follows from the established inequality of the lemma since $\hat{\lambda}(T)$ must be an exterior normal of $\overline{K}(T, x_o)$ at the corresponding response end point, $\hat{x}_{u*}(T)$.

CONCLUSIONS

It has been proven the Maximum Principle is often a sufficient as well as necessary condition for optimal control.

REFERENCES

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2. Neustadt, L. W., "The Existence of Optimal Controls in the Absence of Convexity Condition", Jour. of Math. Ana. and App. (to appear) 1963.